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Adiabatic regularisation for scalar fields with arbitrary coupling to the scalar curvature

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Abstract. Adiabatic regularisation is applied to a scalar field propagating in a Robertson–Walker universe with arbitrary coupling to the scalar curvature. Explicit expressions for the expectation value of the quantum stress tensor in an adiabatic vacuum are obtained. This calculation yields the terms which are to be subtracted from the divergent mode-sum expressions for expectation values of the stress tensor to give a finite, renormalised stress tensor. It is shown that the removal of the infinite terms in this subtraction procedure corresponds to the renormalisation of coupling constants in Einstein's equation. A short description is given of the way in which adiabatic regularisation produces a trace anomaly.

1. Introduction

The recent study of the stress tensor, $T_{\mu\nu}$, of quantum fields in curved space–time has led to the introduction of a number of different regularisation schemes designed to make finite the formally divergent matrix elements, $\langle T_{\mu\nu}\rangle$, of the stress tensor. However, the use of a method of regularisation is only one stage in the process of defining a renormalised stress tensor. First, one must define physically acceptable quantum states so that the stress tensor operator can be studied via its matrix elements. Of particular interest are the *expectation values* of $T_{\mu\nu}$ in states which represent the distribution of quantum matter throughout the space–time since these quantities appear as the source of the gravitational field in Einstein's equation. It is usual to consider only vacuum expectation values, $\langle 0|T_{\mu\nu}|0\rangle$, since these have a simple expression as a formally divergent sum or integral over products of modes and their derivatives. Once a vacuum expectation value has been renormalised by the subtraction of a formally infinite quantity, the renormalised operator $T_{\mu\nu}$ is immediately given by the subtraction of the same infinite quantity. Secondly, in order to carry out this subtraction, the divergent mode sum must be regularised, preferably by introducing some covariant cutoff. Thirdly, the regularised mode sum is decomposed into two parts: an unphysical, divergent part which is to be discarded, and the finite physical remainder, the renormalised vacuum expectation value. Fourthly, some justification for discarding the divergent part should be given. In practice this is usually done by renormalising coupling constants in a generalised form of Einstein's equation which includes geometric tensors which are fourth order in derivatives of the metric.

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The third stage in the renormalisation process is usually carried out in practice by one of two methods, by adiabatic regularisation (Parker and Fulling 1974) or by using the DeWitt–Schwinger formalism (Schwinger 1951, DeWitt 1975). Both of these methods calculate only the divergent part of $\langle T_{\mu\nu}\rangle$ which is to be removed in the renormalisation process. They do not give any detailed information about the finite remainder, which is obtained from the mode sum after the divergences have been identified and discarded. The DeWitt–Schwinger formalism has the advantage of being completely general: the divergences in $\langle T_{\mu\nu}\rangle$ are known for an arbitrary background metric. Moreover, it is known that if dimensional regularisation is used, these divergences can be removed by renormalising coupling constants in Einstein's equation (Bunch 1979). However, the evaluation of the renormalised expectation values of $T_{\mu\nu}$ is rather complicated, although some explicit calculations have been performed using covariant point-splitting regularisation (Bunch and Davies 1978a, b). This method of regularisation does not lead to the renormalisation of coupling constants in Einstein's equation so that the problem of determining the correct renormalisation procedure is less simple than with dimensional regularisation: this matter is discussed in detail in Bunch *et al* (1978). In contrast, adiabatic regularisation has the advantage of being easy to apply to concrete calculations since it enables renormalised expectation values of $T_{\mu\nu}$ to be calculated as finite mode sums (Bunch 1978), although in practice a simple cutoff procedure is sometimes required to evaluate the finite integrals (Birrell 1978). The main disadvantage of adiabatic regularisation is its lack of generality: it has only been developed for scalar fields with particular couplings to the scalar curvature in Robertson–Walker and Kasner space–times (Parker and Fulling 1974, Fulling *et al* 1974). Moreover, an early attempt to show that the divergences in $\langle T_{\mu\nu}\rangle$ which are removed by adiabatic regularisation can renormalise coupling constants in Einstein's equation was only partially successful (Fulling and Parker 1974).

In § 2 of this paper a generalisation of adiabatic regularisation will be given in which it is applied to a scalar field propagating in a Robertson–Walker universe with *arbitrary* coupling to the scalar curvature. The generalisation to arbitrary coupling is important when a self-coupling is included ($\lambda\phi^4$ field theory) because the constant which measures the strength of this coupling undergoes renormalisation in such a theory. An example of the use of adiabatic regularisation in $\lambda\phi^4$ field theory appears in Bunch *et al* (1980). Section 3 of this paper is devoted to showing that the explicitly divergent terms in $\langle T_{\mu\nu}\rangle$ *can* lead to the renormalisation of coupling constants in Einstein's equation provided that a suitable cutoff (method of regularisation) is used. Finally, in § 4 a short discussion is given of the appearance of an anomalous trace in the renormalised stress tensor.

2. Adiabatic regularisation of the stress tensor

The Robertson–Walker metric will be taken in the form:

$$ds^2 = C(\eta)[d\eta^2 - h_{ij}\, dx^i\, dx^j] \qquad (i = 1, 2, 3) \tag{2.1}$$

$$h_{ij}\, dx^i\, dx^j = (1-\epsilon r^2)^{-1}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2.2}$$

where $\epsilon = -1$, 0 or $+1$ for spatially open, flat or closed universes. The scalar wave equation is

$$\Box\phi + (m^2 + \xi R)\phi = 0. \tag{2.3}$$

Put

$$\phi = C^{-1/2}\chi \tag{2.4}$$

and decompose

$$\chi = \int d\tilde{\mu}(\boldsymbol{k})[A_{\boldsymbol{k}} Y_{\boldsymbol{k}}(\boldsymbol{x})\chi_k(\eta) + A_{\boldsymbol{k}}^\dagger Y_{\boldsymbol{k}}^*(\boldsymbol{x})\chi_k^*(\eta)] \tag{2.5}$$

where

$$\Delta^{(3)} Y_{\boldsymbol{k}}(\boldsymbol{x}) \equiv h^{-1/2}\partial_i[h^{1/2}h^{ij}\partial_j Y_{\boldsymbol{k}}(\boldsymbol{x})] = -(k^2 - \epsilon) Y_{\boldsymbol{k}}(\boldsymbol{x}) \tag{2.6}$$

$$k = 1, 2, 3, \ldots \qquad \text{if } \epsilon = 1 \tag{2.7a}$$

$$0 < k < \infty \qquad \text{if } \epsilon = 0 \text{ or } -1 \tag{2.7b}$$

$$h = \det(h_{ij}) \tag{2.8}$$

and

$$\int d\tilde{\mu}(\boldsymbol{k}) = \begin{cases} \int d^3k & \text{if } \epsilon = 0 \\ \sum\limits_{l,m,n} \text{ or } \sum\limits_{l,J,M} & \text{if } \epsilon = 1 \\ \int_0^\infty dk \sum\limits_{J,M} & \text{if } \epsilon = -1. \end{cases} \tag{2.9}$$

The properties of the functions $Y_{\boldsymbol{k}}(\boldsymbol{x})$ are discussed in appendix A of Parker and Fulling (1974). The functions $\chi_k(\eta)$ in (2.5) satisfy:

$$\chi_k'' + \Omega_k^2 \chi_k = 0 \tag{2.10}$$

where

$$\Omega_k^2 = \omega_k^2 + (\xi - \tfrac{1}{6})CR \tag{2.11}$$

$$\omega_k^2 = k^2 + Cm^2 \tag{2.12}$$

and the primes in (2.10) denote differentiation with respect to η. The functions χ_k are normalised according to

$$\chi_k^{*\prime}\chi_k - \chi_k^*\chi_k' = \mathrm{i} \tag{2.13}$$

which ensures that the canonical commutation relations for the field operator ϕ and its conjugate momentum lead to the following commutation relations for the operators $A_{\boldsymbol{k}}$ and $A_{\boldsymbol{k}}^\dagger$:

$$[A_{\boldsymbol{k}}, A_{\boldsymbol{k}'}] = 0 = [A_{\boldsymbol{k}}^\dagger, A_{\boldsymbol{k}'}^\dagger]$$

$$[A_{\boldsymbol{k}}, A_{\boldsymbol{k}'}^\dagger] = \delta(\boldsymbol{k}, \boldsymbol{k}') \tag{2.14}$$

where

$$\int d\tilde{\mu}(\boldsymbol{k}) f(\boldsymbol{k})\, \delta(\boldsymbol{k}, \boldsymbol{k}') = f(\boldsymbol{k}'). \tag{2.15}$$

An adiabatic vacuum $|0\rangle_A$ is now defined by choosing χ_k to be a positive-frequency WKB solution of (2.11) and by taking $|0\rangle_A$ to be a state annihilated by all the operators $A_{\boldsymbol{k}}$. This means that χ_k is taken to be

$$\chi_k = (2W_k)^{-1/2} \exp\left(-\mathrm{i}\int^\eta W_k(\tilde{\eta})\, d\tilde{\eta}\right) \tag{2.16}$$

where the equation satisfied by W_k is obtained by substituting (2.16) in (2.10) which gives:

$$W_k^2=\Omega_k^2-\frac{1}{2}\left(\frac{W_k''}{W_k}-\frac{3}{2}\frac{W_k'^2}{W_k^2}\right). \tag{2.17}$$

The WKB solution is obtained by solving (2.17) iteratively, taking the zeroth-order WKB solution to be

$$W_k^{(0)}=\Omega_k. \tag{2.18}$$

The first iterated WKB solution is

$$W_k^{(1)2}=\Omega_k^2-\frac{1}{2}\left(\frac{W_k^{(0)''}}{W_k^{(0)}}-\frac{3}{2}\frac{W_k^{(0)'2}}{W_k^{(0)2}}\right). \tag{2.19}$$

It is not difficult to see that higher-order WKB solutions contain terms involving increasingly many derivatives with respect to η. To obtain all the divergences in $\langle T_{\mu\nu}\rangle$, it is sufficient to calculate W_k to an order which includes all terms involving no more than four derivatives with respect to η. Such terms are referred to as terms of adiabatic order T^{-4}. This is discussed in considerable detail in Parker and Fulling (1974). The reason why the WKB solution to order T^{-4} yields the divergences in $\langle T_{\mu\nu}\rangle$ is that the WKB solution is an asymptotic solution in large ω_k, or equivalently in large k. Thus higher-order terms fall off sufficiently rapidly as $k\to\infty$ to give a finite contribution to $\langle T_{\mu\nu}\rangle$. Since the WKB approximation is valid for large k, an adiabatic vacuum has the physical interpretation of representing a distribution of quantum matter which, to a given order in k, is vacuous in the high-frequency modes.

Having determined $W_k^{(1)}$ to order T^{-4} from (2.19), a second iteration yields $W_k^{(2)}$ to order T^{-4}. Further iterations only yield terms of higher adiabatic order so one obtains the following result (which could also have been obtained by taking $W_k^{(0)}=\omega_k$ in place of (2.18)):

$$\begin{aligned}W\approx\omega&+\frac{(\xi-\frac{1}{6})}{4\omega}(6D'+3D^2+12\epsilon)-\frac{Cm^2}{8\omega^3}(D'+D^2)+\frac{5C^2m^4D^2}{32\omega^5}\\&+\frac{Cm^2}{32\omega^5}(D'''+4D''D+3D'^2+6D'D^2+D^4)\\&-\frac{C^2m^4}{128\omega^7}(28D''D+19D'^2+122D'^2+47D^4)\\&+\frac{221C^3m^6}{256\omega^9}(D'D^2+D^4)-\frac{1105C^4m^8D^4}{2048\omega^{11}}\\&-\frac{(\xi-\frac{1}{6})}{8\omega^3}(3D'''+3D''D+3D'^2)\\&+(\xi-\tfrac{1}{6})\frac{Cm^2}{32\omega^5}(30D''D+18D'^2+57D'D^2+9D^4+36D'\epsilon+36D^2\epsilon)\\&-(\xi-\tfrac{1}{6})\frac{75C^2m^4}{128\omega^7}(2D'D^2+D^4+4D^2\epsilon)\\&-\frac{(\xi-\frac{1}{6})^2}{32\omega^3}(36D'^2+36D'D^2+9D^4+144D'\epsilon+72D^2\epsilon+144\epsilon^2)\end{aligned} \tag{2.20}$$

where $D = C'/C$, the subscript k has been omitted from W and ω, and the Ricci scalar has been taken to be

$$R = C^{-1}(3D' + \tfrac{3}{2}D^2 + 6\epsilon). \tag{2.21}$$

Notice that this indicates that ϵ is of adiabatic order T^{-2}. Consider now the stress tensor which for arbitrary ξ has the classical form:

$$T_{\mu\nu} = (1-2\xi)\partial_\mu\phi\partial_\nu\phi + (2\xi - \tfrac{1}{2})g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi - 2\xi\phi\nabla_\mu\partial_\nu\phi + 2\xi g_{\mu\nu}\phi\Box\phi - \xi G_{\mu\nu}\phi^2 + \tfrac{1}{2}m^2 g_{\mu\nu}\phi^2. \tag{2.22}$$

With the metric (2.1), one obtains

$$T_{00} = \tfrac{1}{2}(\partial_0\phi)^2 + (\tfrac{1}{2} - 2\xi)h^{ij}\partial_i\phi\partial_j\phi + 3\xi D\phi\partial_0\phi - 2\xi\phi\Delta^{(3)}\phi + \tfrac{3}{4}\xi(D^2 + 4\epsilon)\phi^2 + \tfrac{1}{2}Cm^2\phi^2. \tag{2.23}$$

It was argued by Fulling *et al* (1974) that for a state representing a distribution of matter having the usual Robertson–Walker symmetries, $\langle T_{00}\rangle$ can be replaced by

$$\langle T_{00}\rangle = \left(\int d^3\boldsymbol{x}\,\sqrt{h}\right)^{-1}\left\langle\int d^3\boldsymbol{x}\,\sqrt{h}\,T_{00}\right\rangle. \tag{2.24}$$

When $\epsilon = 0$ or -1, the right-hand side of (2.24) is to be interpreted as the limit of a ratio of quantities integrated over a large finite region of space–time. Applying this relation to the second term in (2.23) leads to

$$\begin{aligned}\int d^3\boldsymbol{x}\,\sqrt{h}\,h^{ij}\partial_i\phi\partial_j\phi &= \int d^3\boldsymbol{x}\,\partial_i(\sqrt{h}\,h^{ij}\phi\partial_j\phi) - \int d^3\boldsymbol{x}\,\phi\partial_i(\sqrt{h}\,h^{ij}\partial_j\phi)\\ &= \int d^3\boldsymbol{x}\,\sqrt{h}\,\nabla_i(\phi\partial^i\phi) - \int d^3\boldsymbol{x}\,\sqrt{h}\,\phi\Delta^{(3)}\phi\end{aligned} \tag{2.25}$$

where ∇_i is the covariant derivative on the three-space with metric h_{ij}. The first term can be converted to a surface term which gives no contribution to $\langle T_{00}\rangle$ since it is assumed that expectation values of field operators fall off rapidly at large spatial distances. Hence:

$$\langle T_{00}\rangle = \tfrac{1}{2}\langle(\partial_0\phi)^2\rangle - \tfrac{1}{2}\langle\phi\Delta^{(3)}\phi\rangle + 3\xi D\langle\phi\partial_0\phi\rangle + 3\xi(\tfrac{1}{4}D^2 + \epsilon)\langle\phi^2\rangle + \tfrac{1}{2}Cm^2\langle\phi^2\rangle. \tag{2.26}$$

Now use (2.4)–(2.6) and the following relation from Fulling *et al* (1974, equation (5.21)):

$$\int d\tilde{\mu}(\boldsymbol{k})|Y_{\boldsymbol{k}}(x)|^2 f(k) = (2\pi^2)^{-1}\int d\mu(k)f(k) \tag{2.27}$$

where

$$\int d\mu(k) = \begin{cases}\displaystyle\int_0^\infty k^2\,dk & \text{if } \epsilon = 0 \text{ or } -1\\ \displaystyle\sum_{k=1}^{\infty} k^2 & \text{if } \epsilon = 1.\end{cases} \tag{2.28}$$

This yields:

$$\langle T_{00}\rangle = (4\pi^2 C)^{-1} \int d\mu(k)\{|\psi_k'|^2 + \omega_k^2|\psi_k|^2 + (\xi - \tfrac{1}{6})[3D(\psi_k\psi_k^{*\prime} + \psi_k^*\psi_k') - \tfrac{3}{2}(D^2 - 4\epsilon)|\psi_k|^2]\} \tag{2.29}$$

where an operator symmetrisation of the term $\langle\phi\partial_0\phi\rangle$ has been performed. Expressions for $|\psi_k|^2$, $|\psi_k'|^2$ and $\psi_k\psi_k^{*\prime} + \psi_k^*\psi_k'$ calculated to order T^{-4} are given in appendix 1. Using these, ${}_A\langle 0|T_{00}|0\rangle_A$ can be calculated and the following result obtained:

$$\begin{aligned} {}_A\langle 0|T_{00}|0\rangle_A = (8\pi^2 C)^{-1} \int d\mu(k)\Big[& 2\omega + \frac{C^2m^4D^2}{16\omega^5} - \frac{C^2m^4}{64\omega^7}(2D''D - D'^2 + 4D'D^2 + D^4) \\ & + \frac{7C^3m^6}{64\omega^9}(D'D^2 + D^4) - \frac{105C^4m^8D^4}{1024\omega^{11}} \\ & + (\xi - \tfrac{1}{6})\Big(-\frac{3}{2\omega}(D^2 - 4\epsilon) - \frac{3Cm^2D^2}{2\omega^3} + \frac{Cm^2}{8\omega^5}(6D''D - 3D'^2 + 6D'D^2) \\ & - \frac{C^2m^4}{64\omega^7}(120D'D^2 + 105D^4 + 60D^2\epsilon) + \frac{105C^3m^6D^4}{64\omega^9}\Big) \\ & + (\xi - \tfrac{1}{6})^2\Big(-\frac{1}{16\omega^3}(72D''D - 36D'^2 - 27D^4 - 72D^2\epsilon + 144\epsilon^2) \\ & + \frac{Cm^2}{8\omega^5}(54D'D^2 + 27D^4 + 108D^2\epsilon)\Big)\Big]. \end{aligned} \tag{2.30}$$

The other independent component of $\langle T_{\mu\nu}\rangle$, namely $\langle T_{11}\rangle$, can be obtained most simply from (2.30) and the trace of $\langle T_{\mu\nu}\rangle$. No renormalisation subtractions have yet been made, so this procedure is not in conflict with the eventual appearance of a trace anomaly in the renormalised stress tensor. Making use of the wave equation (2.3), the trace of the stress tensor operator may be written:

$$T_\alpha^\alpha = (6\xi - 1)\partial^\alpha\phi\partial_\alpha\phi + \xi(1 - 6\xi)R\phi^2 + 2(1 - 3\xi)m^2\phi^2. \tag{2.31}$$

The component T_{11} is related to the trace and T_{00} by

$$T_{11} = \tfrac{1}{3}\rho(T_{00} - CT_\alpha^\alpha) \tag{2.32}$$

where

$$\rho \equiv (1 - \epsilon r^2)^{-1}. \tag{2.33}$$

Using (2.24), (2.25), (2.4)–(2.6) amd (2.27) the following expression for $\langle T_\alpha^\alpha\rangle$ is soon obtained:

$$\langle T_\alpha^\alpha\rangle = (2\pi^2C^2)^{-1} \int d\mu(k)\{Cm^2|\psi_k|^2 + 6(\xi - \tfrac{1}{6})[|\psi_k'|^2 - \tfrac{1}{2}D(\psi_k\psi_k^{*\prime} + \psi_k^*\psi_k') - \omega_k^2|\psi_k|^2 - \tfrac{1}{2}D'|\psi_k|^2 - (\xi - \tfrac{1}{6})(3D' + \tfrac{3}{2}D^2 + 6\epsilon)|\psi_k|^2]\}. \tag{2.34}$$

Evaluating this using the expressions in appendix 1 leads to

$$
\begin{aligned}
{}_A\langle 0|T^\alpha_\alpha|0\rangle_A = (4\pi^2C^2)^{-1}\int \mathrm{d}\mu(k)\Big[&\frac{Cm^2}{\omega}+\frac{C^2m^4}{8\omega^5}(D'+D^2)-\frac{5C^3m^6D^2}{32\omega^7}\\
&-\frac{C^2m^4}{32\omega^7}(D'''+4D''D+3D'^2+6D'D^2+D^4)\\
&+\frac{C^3m^6}{128\omega^9}(28D''D+21D'^2+126D'D^2+49D^4)-\frac{231C^4m^8}{256\omega^{11}}(D'D^2+D^4)\\
&+\frac{1155C^5m^{10}D^4}{2048\omega^{13}}+(\xi-\tfrac{1}{6})\Big(-\frac{3D'}{\omega}-\frac{Cm^2}{\omega^3}(3D'+\tfrac{3}{4}D^2+3\epsilon)\\
&+\frac{9C^2m^4D^2}{4\omega^5}+\frac{Cm^2}{4\omega^5}(3D'''+6D''D+\tfrac{9}{2}D'^2+3D'D^2)\\
&-\frac{C^2m^4}{32\omega^7}(120D''D+90D'^2+390D'D^2+105D^4+60D'\epsilon+60D^2\epsilon)\\
&+\frac{C^3m^6}{128\omega^9}(1680D'D^2+1365D^4+420D^2\epsilon)-\frac{945C^4m^8D^4}{128\omega^{11}}\Big)\\
&+(\xi-\tfrac{1}{6})^2\Big(-\frac{1}{4\omega^3}(18D'''-27D'D^2-36D'\epsilon)\\
&+\frac{Cm^2}{32\omega^5}(432D''D+324D'^2+648D'D^2\\
&+27D^4+864D'\epsilon+216D^2\epsilon+432\epsilon^2)\\
&-\frac{C^2m^4}{16\omega^7}(270D'D^2+135D^4+540D^2\epsilon)\Big)\Big].
\end{aligned}
\tag{2.35}
$$

An expression for ${}_A\langle 0|T_{11}|0\rangle_A$ as an integral over k can be obtained from (2.30) and (2.35) using (2.32).

So far, nothing has been said about regularisation. In spite of its name, adiabatic regularisation is not a method of regularising divergent integrals. Thus the expressions (2.30) and (2.35), which are the main results of this paper, consist of formally divergent integrals and, in principle, some covariant cutoff should be introduced to make sense of them. Once this has been done, it is possible to discuss carefully how to define renormalised matrix elements of $T_{\mu\nu}$. There are two covariant methods that may be used to regularise ${}_A\langle 0|T_{\mu\nu}|0\rangle_A$: covariant point-splitting and dimensional regularisation. When covariant point-splitting is used, ${}_A\langle 0|T_{\mu\nu}|0\rangle_A$ is obtained as a function of two points, x and x'. It can also be expressed as a function of x, ϵ and t^μ where x' is situated an affine parameter distance ϵ along the geodesic which has unit tangent vector t^μ at x. A discussion of how to define the renormalised stress tensor from the expression for ${}_A\langle 0|T_{\mu\nu}|0\rangle_A$ as a function of x, ϵ and t^μ was given in Bunch *et al* (1978). The procedure proposed there is to discard all terms which depend explicitly on the regularisation parameters ϵ and t^μ and, in addition, to discard any local geometrical quantities whose presence would otherwise prevent the renormalised matrix element from being conserved. This determines the renormalised expectation value up to multiples of conserved geometric tensors of adiabatic order up to T^{-4}. There are four such tensors: the metric $g_{\mu\nu}$, Einstein's tensor $G_{\mu\nu}$, and the tensors ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ which are obtained by varying $\int R^2\sqrt{g}\,d^4x$ and $\int R_{\alpha\beta}R^{\alpha\beta}\sqrt{g}\,d^4x$ with respect to the metric, where g is the determinant of the metric. The coefficients of these tensors are presumably to be determined by experiment, as in any renormalised field theory. This renormalisation procedure was shown in Bunch *et al* (1978) to yield a renormalised stress tensor satisfying the first four of the axioms of Wald (1977). In addition, it was shown that for a massive scalar field, the procedure is equivalent to discarding *all* the terms appearing in ${}_A\langle 0|T_{\mu\nu}|0\rangle_A$ which are of adiabatic order up to T^{-4}, whether they depend on the regularisation parameters or not. This last point is important because it means that one can calculate renormalised expectation values of $T_{\mu\nu}$ by writing down a formal mode-sum expression for the expectation value and subtracting from this the quantity ${}_A\langle 0|T_{\mu\nu}|0\rangle_A$ which is given by (2.30) and (2.35). This subtraction can be performed mode by mode leaving *finite* integrals which can be evaluated without having to introduce a covariant cutoff. Thus, although one needs to use covariant regularisation to justify the renormalisation procedure being used, one can perform practical calculations without it. Indeed, it is even possible to perform these mode by mode subtractions for *massless* scalar fields provided that one starts out with non-zero mass and only takes the limit $m \to 0$ after the renormalised $\langle T_{\mu\nu}\rangle$ has been obtained (see, for example, Bunch 1978).

To summarise the conclusions of this section: expectation values of $T_{\mu\nu}$ are calculated from formally divergent mode sums. An adiabatic vacuum state, $|0\rangle_A$, can be defined using a WKB approximation which characterises the high-frequency behaviour of the quantum field. Thus the divergences in the original mode sum, which come from the high-frequency modes, are the same as the divergences in ${}_A\langle 0|T_{\mu\nu}|0\rangle_A$. A careful analysis of the structure of ${}_A\langle 0|T_{\mu\nu}|0\rangle_A$ carried out using covariant point-splitting to regularise the formally divergent integrals shows that the original mode sum can be renormalised by subtracting from it the quantity ${}_A\langle 0|T_{\mu\nu}|0\rangle_A$ calculated to adiabatic order T^{-4}. This subtraction can be applied to the integrand of the original mode sum, leaving finite integrals. The resulting expression is conserved and determines the renormalised expectation value up to multiples of the conserved geometrical tensors $g_{\mu\nu}$, $G_{\mu\nu}$, ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$.

3. Renormalisation of coupling constants in Einstein's equation

Covariant point-splitting is not a suitable method of regularisation to use when discussing whether the removal of divergences from $\langle T_{\mu\nu}\rangle$ can be carried out by renormalising coupling constants in Einstein's equation. This is partly because $\langle T_{\mu\nu}\rangle$ depends on the vector t^μ, but even if this dependence were removed (say, by averaging over all directions), coupling constant renormalisation would still not be possible since covariant regularisation methods which operate in four dimensions cannot give a trace anomaly and at the same time renormalise coupling constants (Bunch 1979). Thus one must use dimensional regularisation. To perform a proper dimensional regularisation would involve carrying out the calculations of § 2 entirely in n dimensions. To avoid such a complicated calculation, a simple *non-covariant* 'dimensional regularisation' will be used instead. This regularisation consists of replacing k^2 by k^{n-2} in (2.28). Because this procedure is not covariant, it is not possible to show that renormalising coupling constants in Einstein's equation removes *all* terms of adiabatic order up to T^{-4}. Instead, it will be shown that the explicitly divergent terms are removed in this way.

Consider first the terms of adiabatic order zero:

$$\langle T_{00}\rangle^{(0)} = \frac{1}{4\pi^2 C}\int_0^\infty \omega_k k^{n-2}\,\mathrm{d}k \tag{3.1}$$

$$\langle T_{11}\rangle^{(0)} = \frac{\rho}{12\pi^2 C}\int_0^\infty \frac{k^n\,\mathrm{d}k}{\omega_k}. \tag{3.2}$$

Performing the integrations and retaining only the pole at $n=4$:

$$\langle T_{00}\rangle^{(0)} \approx \frac{Cm^4}{32\pi^2(n-4)} \tag{3.3}$$

$$\langle T_{11}\rangle^{(0)} \approx -\frac{C\rho m^4}{32\pi^2(n-4)} \tag{3.4}$$

which implies

$$\langle T_{\mu\nu}\rangle^{(0)} \approx \frac{m^4 g_{\mu\nu}}{32\pi^2(n-4)}. \tag{3.5}$$

Thus this quantity can be removed by renormalising the cosmological constant.

Now consider the explicitly divergent terms of adiabatic order T^{-2}:

$$\langle T_{00}\rangle^{(2)}_{\text{div}} = -\frac{3(\xi-\frac{1}{6})}{16\pi^2 C}(D^2-4\epsilon)\int_0^\infty \frac{k^{n-2}\,\mathrm{d}k}{\omega_k} - \frac{3(\xi-\frac{1}{6})m^2 D^2}{16\pi^2}\int_0^\infty \frac{k^{n-2}\,\mathrm{d}k}{\omega_k^3} \tag{3.6}$$

$$\approx \frac{3(\xi-\frac{1}{6})m^2}{32\pi^2(n-4)}(D^2+4\epsilon) = -\frac{(\xi-\frac{1}{6})m^2}{8\pi^2(n-4)}G_{00} \tag{3.7}$$

$$\langle T_{11}\rangle^{(2)}_{\text{div}} = \frac{(\xi-\frac{1}{6})\rho}{48\pi^2 C}\left((12D'-3D^2+12\epsilon)\int_0^\infty \frac{k^{n-2}\,\mathrm{d}k}{\omega_k} + Cm^2(12D'+12\epsilon)\int_0^\infty \frac{k^{n-2}\,\mathrm{d}k}{\omega_k^3}\right) \tag{3.8}$$

$$\approx -\frac{m^2(\xi-\frac{1}{6})}{48\pi^2(n-4)}\rho(6D'+\tfrac{3}{2}D^2+6\epsilon) = -\frac{(\xi-\frac{1}{6})m^2}{8\pi^2(n-4)}G_{11}. \tag{3.9}$$

Thus

$$\langle T_{\mu\nu}\rangle^{(2)}_{\text{div}} \approx -\frac{(\xi-\frac{1}{6})m^2 G_{\mu\nu}}{8\pi^2(n-4)}. \tag{3.10}$$

This can be removed by renormalising the gravitational constant.

It is not necessary to use any regularisation to investigate the divergent terms of adiabatic order T^{-4} since the coefficients of these terms are already the components of the conserved tensor ${}^{(1)}H_{\mu\nu}$:

$$\langle T_{00}\rangle^{(4)}_{\text{div}} = -\frac{(\xi-\frac{1}{6})^2}{128\pi^2 C}(72D''D-36D'^2-27D^4-72D^2\epsilon+144\epsilon^2)\int_0^\infty \frac{k^2\,\mathrm{d}k}{\omega_k^3} \tag{3.11}$$

$$\langle T_{11}\rangle^{(4)}_{\text{div}} = \frac{(\xi-\frac{1}{6})^2}{128\pi^2 C}\rho(48D'''-24D''D+12D'^2-72D'D^2+9D^4$$

$$-96D'\epsilon+24D^2\epsilon-48\epsilon^2)\int_0^\infty \frac{k^2\,\mathrm{d}k}{\omega_k^3}. \tag{3.12}$$

These lead to

$$\langle T_{\mu\nu}\rangle^{(4)}_{\text{div}} = \frac{(\xi-\frac{1}{6})^2}{16\pi^2}\,{}^{(1)}H_{\mu\nu}\int_0^\infty \frac{k^2\,\mathrm{d}k}{\omega_k^3}. \tag{3.13}$$

This completes the demonstration that the explicitly infinite terms in $\langle T_{\mu\nu}\rangle$ can be removed by renormalising coupling constants in Einstein's equation. The discussion given at the end of § 2 indicates that the explicitly finite terms in (2.30) and (2.35) must also be subtracted from $\langle T_{\mu\nu}\rangle$ in the renormalisation process. These terms are not covariant, reflecting the non-covariance of adiabatic regularisation. However, as discussed earlier, if adiabatic regularisation were developed entirely in n dimensions, this covariant regularisation would give a completely covariant result.

The finite terms in the adiabatic stress tensor can be explicitly evaluated for $\epsilon = 0$ or -1 by performing the integrations. The result is

$$\begin{aligned}\langle T_{00}\rangle_{\text{finite}} &= \frac{m^2D^2}{384\pi^2} - \frac{1}{2880\pi^2C}(\tfrac{3}{2}D''D - \tfrac{3}{4}D'^2 - \tfrac{3}{8}D^4)\\ &\quad + \frac{(\xi-\frac{1}{6})}{256\pi^2C}(8D''D - 4D'^2 - 3D^4 - 4D^2\epsilon)\\ &\quad + \frac{(\xi-\frac{1}{6})^2}{64\pi^2C}(18D'D^2 + 9D^4 + 36D^2\epsilon)\end{aligned} \tag{3.14}$$

$$\begin{aligned}\langle T_{11}\rangle_{\text{finite}} &= \rho\Big[-\frac{m^2}{1152\pi^2}(4D' + D^2 + 72(\xi-\tfrac{1}{6})D^2)\\ &\quad + \frac{1}{2880\pi^2C}\left(D''' - \tfrac{1}{2}D''D + \tfrac{1}{4}D'^2 - D'D^2 + \tfrac{1}{8}D^4\right)\\ &\quad - \frac{(\xi-\frac{1}{6})}{768\pi^2C}(16D''' - 8D''D + 4D'^2 - 24D'D^2 + 3D^4 - 16D'\epsilon + 4D^2\epsilon)\\ &\quad - \frac{(\xi-\frac{1}{6})^2}{192\pi^2C}(72D''D + 54D'^2 + 54D'D^2 - \tfrac{45}{2}D^4 + 144D'\epsilon\\ &\quad - 72D^2\epsilon + 72\epsilon^2)\Big].\end{aligned} \tag{3.15}$$

When $\xi = 0$ or $\frac{1}{6}$ and $\epsilon = 0$, the terms of adiabatic order T^{-4} agree with those obtained by Birrell (1978). To check this requires realising that Birrell's logarithmic divergence is essentially

$$\int_0^\infty \frac{\mathrm{d}k}{\omega_k} \equiv \int_0^\infty \frac{k^2\,\mathrm{d}k}{\omega_k^3} + Cm^2\int_0^\infty \frac{\mathrm{d}k}{\omega_k^3} \tag{3.16}$$

$$= \int_0^\infty \frac{k^2\,\mathrm{d}k}{\omega_k^3} + 1. \tag{3.17}$$

Thus expressions (3.14) and (3.15) above contain not only Birrell's finite term but also a finite contribution from his logarithmic divergence.

4. The trace anomaly

The trace of the classical stress tensor (2.22) is

$$T^\alpha_\alpha = (6\xi - 1)\partial^\alpha\phi\partial_\alpha\phi + 6\xi\phi\Box\phi + (\xi R + 2m^2)\phi^2. \tag{4.1}$$

Using the wave equation this may be expressed in a form that is manifestly traceless when $\xi = \frac{1}{6}$ and $m = 0$:

$$T^\alpha_\alpha = (6\xi - 1)\partial^\alpha\phi\partial_\alpha\phi + (6\xi - 1)\phi\Box\phi + m^2\phi^2. \tag{4.2}$$

It is well known that, although this trace is formally zero for a massless conformally coupled scalar field, its renormalised expectation values are not necessarily zero. This arises because renormalisation of the trace by adiabatic regularisation requires the subtraction of the massless limit of $m^2{}_A\langle 0|\phi^2|0\rangle_A$ which is non-zero since ${}_A\langle 0|\phi^2|0\rangle_A$ contains finite terms of adiabatic order T^{-4} which are proportional to m^{-2}. This mechanism gives rise to a trace anomaly even when $\xi \neq \frac{1}{6}$ since an unexpected contribution to the renormalised trace of the massless theory is produced in the manner just described.

Thus the trace anomaly provided by adiabatic regularisation is

$$\langle T^\alpha_\alpha\rangle_{\text{anomalous}} = -\lim_{m\to 0} m^2{}_A\langle 0|\phi^2|0\rangle_A \tag{4.3}$$

$$= -\lim_{m\to 0}\frac{m^2}{4\pi^2 C}\int_0^\infty \frac{k^2\,\mathrm{d}k}{W_k} \tag{4.4}$$

$$= \frac{1}{960\pi^2 C^2}(D''' - D'D^2) - \frac{(\xi - \frac{1}{6})}{64\pi^2 C^2}(2D''' - 3D'D^2 - 4D'\epsilon)$$
$$- \frac{(\xi - \frac{1}{6})^2}{128\pi^2 C^2}(36D'^2 + 36D'D^2 + 9D^4 + 144D'\epsilon + 72D^2\epsilon + 144\epsilon^2) \tag{4.5}$$

$$= \frac{1}{2880\pi^2}[\Box R - (R^{\alpha\beta}R_{\alpha\beta} - \tfrac{1}{3}R^2)] - \frac{(\xi - \frac{1}{6})}{96\pi^2}\Box R - \frac{(\xi - \frac{1}{6})^2 R^2}{32\pi^2}. \tag{4.6}$$

This expression for the trace anomaly is of the general form derived using dimensional regularisation by Deser *et al* (1976) and is equal to $-a_2(x)/16\pi^2$ where $a_2(x)$ is a coefficient which arises in the DeWitt–Schwinger formalism. It is important to realise that, except when $\xi = \frac{1}{6}$, when everything that appears in the renormalised trace is anomalous, there is some ambiguity about precisely how much of the non-zero trace should be regarded as anomalous. (There is, of course, no ambiguity about the entire renormalised trace.) This ambiguity arises because the functional form of the stress tensor can be altered by using the wave equation. Different contributions to the anomalous part of the trace are obtained by using each of the three expressions (2.31), (4.1) and (4.2) because the coefficient of $m^2\phi^2$ is different in each case. This ambiguity only alters the trace anomaly by a multiplicative constant: it always remains proportional to $a_2(x)$.

On the other hand, there is no ambiguity in the following anomalous behaviour which is valid for all ξ. There exists a bilinear operator which is formally zero for a massless field but which has non-zero renormalised matrix elements, namely

$$\Omega \equiv \tfrac{1}{2}(\Box\phi\phi + \phi\Box\phi) + \xi R\phi^2. \tag{4.7}$$

The renormalisation of expectation values of Ω requires the subtraction of a term which is the massless limit of

$$_A\langle 0|\Omega|0\rangle_A \equiv -m^2\,{}_A\langle 0|\phi^2|0\rangle_A. \tag{4.8}$$

Thus the renormalised expectation value of (4.12), in any quantum state, is

$$\langle\Omega\rangle_{\text{ren}} = \lim_{m\to 0} m^2\,{}_A\langle 0|\phi^2|0\rangle_A \tag{4.9}$$

$$= \frac{a_2(x)}{16\pi^2}. \tag{4.10}$$

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Appendix 1. Quantities used in evaluation of the stress tensor

$$|\psi_k|^2 = (2W_k)^{-1} \tag{A1.1}$$

$$\begin{aligned}
W_k^{-1} \approx{}& \frac{1}{\omega} + \frac{Cm^2}{8\omega^5}(D' + D^2) - \frac{5C^2m^4D^2}{32\omega^7} - \frac{(\xi-\frac{1}{6})}{4\omega^3}(6D' + 3D^2 + 12\epsilon) \\
&- \frac{Cm^2}{32\omega^7}(D''' + 4D''D + 3D'^2 + 6D'D^2 + D^4) \\
&+ \frac{C^2m^4}{128\omega^9}(28D''D + 21D'^2 + 126D'D^2 + 49D^4) \\
&- \frac{231C^3m^6}{256\omega^{11}}(D'D^2 + D^4) + \frac{1155C^4m^8D^4}{2048\omega^{13}} \\
&+ \frac{(\xi-\frac{1}{6})}{8\omega^5}(3D''' + 3D''D + 3D'^2) \\
&- (\xi-\tfrac{1}{6})\frac{Cm^2}{32\omega^7}(30D''D + 30D'^2 + 75D'D^2 + 15D^4 + 60D'\epsilon + 60D^2\epsilon) \\
&+ (\xi-\tfrac{1}{6})\frac{C^2m^4}{128\omega^9}(210D'D^2 + 105D^4 + 420D^2\epsilon) \\
&+ \frac{(\xi-\frac{1}{6})^2}{32\omega^5}(108D'^2 + 108D'D^2 + 27D^4 + 432D'\epsilon + 216D^2\epsilon + 432\epsilon^2)
\end{aligned} \tag{A1.2}$$

$$\begin{aligned}
\psi_k'\psi_k'^* + \psi_k^*\psi_k' &= -W_k^{-2}W_k' \\
&\approx -\frac{Cm^2D}{2\omega^3} + \frac{Cm^2}{8\omega^5}(D'' + 3D'D + D^3) - \frac{5C^2m^4}{8\omega^7}(D'D + D^3)
\end{aligned}$$

$$+\frac{35C^3m^6D^3}{64\omega^9}-\frac{(\xi-\frac{1}{6})}{2\omega^3}(3D''+3D'D)$$

$$+(\xi-\tfrac{1}{6})\frac{Cm^2}{8\omega^5}(18D'D+9D^3+36D\epsilon) \tag{A1.3}$$

$$|\psi_k'|^2=\frac{1}{4}\frac{W_k'^2}{W_k^3}+W_k$$

$$\approx\omega-\frac{Cm^2}{8\omega^3}(D'+D^2)+\frac{7C^2m^4D^2}{32\omega^5}+\frac{(\xi-\frac{1}{6})}{4\omega}(6D'+3D^2+12\epsilon)$$

$$+\frac{Cm^2}{32\omega^5}(D'''+4D''D+3D'^2+6D'D^2+D^4)$$

$$-\frac{C^2m^4}{128\omega^7}(32D''D+19D'^2+134D'D^2+51D^4)$$

$$+\frac{259C^3m^6}{256\omega^9}(D'D^2+D^4)-\frac{1365C^4m^8D^4}{2048\omega^{11}}$$

$$-\frac{(\xi-\frac{1}{6})}{8\omega^3}(3D'''+3D''D+3D'^2)$$

$$+(\xi-\tfrac{1}{6})\frac{Cm^2}{32\omega^5}(42D''D+18D'^2+69D'D^2+9D^4+36D'\epsilon+36D^2\epsilon)$$

$$-(\xi-\tfrac{1}{6})\frac{105C^2m^4}{128\omega^7}(2D'D^2+D^4+4D^2\epsilon)$$

$$-\frac{(\xi-\frac{1}{6})^2}{32\omega^3}(36D'^2+36D'D^2+9D^4+144D'\epsilon+72D^2\epsilon+144\epsilon^2). \tag{A1.4}$$

Appendix 2. Some geometrical tensors in Robertson–Walker universes

The metric is given by (2.1) and (2.2). There are only two independent components of each two-index tensor in a Robertson–Walker universe, since off-diagonal elements vanish and the space–space diagonal components are proportional to each other. The Ricci tensor and Ricci scalar are

$$R_{00}=\tfrac{3}{2}D' \qquad R_{11}=-\tfrac{1}{2}\rho(D'+D^2+4\epsilon) \tag{A2.1}$$

$$R=C^{-1}(3D'+\tfrac{3}{2}D^2+6\epsilon). \tag{A2.2}$$

Quantities of adiabatic order T^{-4} are, in addition to R^2,

$$\Box R=C^{-2}(3D'''-\tfrac{9}{2}D'D^2-6D'\epsilon) \tag{A2.3}$$

$$R^{\alpha\beta}R_{\alpha\beta}=C^{-2}(3D'^2+\tfrac{3}{2}D'D^2+\tfrac{3}{4}D^4+6D'\epsilon+6D^2\epsilon+12\epsilon^2) \tag{A2.4}$$

$${}^{(1)}H_{00}=C^{-1}(-9D''D+\tfrac{9}{2}D'^2+\tfrac{27}{8}D^4+9D^2\epsilon-18\epsilon^2) \tag{A2.5}$$

$${}^{(1)}H_{11}=C^{-1}\rho(6D'''-3D''D+\tfrac{3}{2}D'^2-9D'D^2+\tfrac{9}{8}D^4-12D'\epsilon+3D^2\epsilon-6\epsilon^2). \tag{A2.6}$$

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